Mathematical Methods of Quantum Mechanics

PhD course in Mathematics @ Unipi

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Course description

Quantum mechanics is one of the central theories of physics, whose language describes a huge variety of systems, from elementary particles to superconducting materials and quantum computers. Fascinating mathematical problems arise in this context, posing hard challenges in functional analysis, partial differential equations and operator theory. This course is an introduction to the rigorous mathematical framework of quantum mechanics and functional analytic methods for the study of the Schrödinger equation.

The course consists of two parts. The first part (of approximately 20 hours) will start with an overview on the theory of linear operators on Hilbert spaces and self-adjointness. We will then study properties of Schrödinger operators and quantum dynamics. Some selected topics related to the physical motivations of quantum mechanics will be presented.

The second part (of approximately 10 hours) will be devoted to many-body quantum mechanics. We will introduce the mathematical questions and tools in this context and conclude with a discussion of current research topics.

Class requirements

No previous knowledge of quantum mechanics is expected, but a solid background in analysis will be necessary to have a deep undestanding of the topics of this course. The class is open to master's students (for master's students the course will grant 6 CFU).

Program of the course

(It may be subjected to small changes)

First part: the mathematical framework of quantum mechanics

1. The formalism of quantum mechanics, Hilbert spaces, Riesz representation theorem.
2. Bounded operators, compact operators. Axioms of Quantum Mechanics.
3. *Why quantum mechanics? Stern-Gerlach experiment. Description of the spin.*
4. Unbounded operators, densely defined operators, closed operators, closed graph theorem, extensions, adjoint operator, symmetric and self-adjoint operators, Hellinger-Töplitz theorem. Momentum operator and multiplication operators.
5. Resolvent, spectrum, analytic functions, operator-valued analytic functions, Neumann series, generalized theorem of the bounded inverse, self-adjointness criterion.

6. *Harmonic oscillator*
7. Kato-Rellich theorem, Schrödinger equation, essential self-adjointness, Schrödinger operators, unitary equivalence, Fourier transform.
8. Weyl criterion, Laplace operators, Sobolev inequalities.
9. *Bell inequality, EPR paradox.*
10. Uniform boundedness principle, strong convergence, strongly continuous unitary groups and generators.
11. Symmetries, quantum Noether theorem.
12. *Discussion of Exercise sheets.*

Second part: many-body quantum mechanics.

1. Confined systems, thermodynamic limit, tensor product of Hilbert spaces, unitary representation of the symmetric group, symmetric and antisymmetric tensor product. Compact operators, trace-class and Hilbert-Schmidt operators, integral kernels.
2. Hamiltonians with pair interaction, Fock space, creation and annihilation operators, canonical (anti)commutation relations
3. Second quantization, reduced density matrix, Bose-Einstein condensation
4. Bogoliubov theory/derivation of the nonlinear Schrödinger equation

References

Main references

- Gerald Teschl: Mathematical Methods in Quantum Mechanics, With Applications to Schrödinger Operators
`https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/index.html`
- Jan Philip Solovej: Many Body Quantum Mechanics, Draft of Lecture Notes of March 5, 2014
`https://web.math.ku.dk/~solovej/MANYBODY/mbnotes-ptn-5-3-14.pdf`

Other references

- Hamiltonian mechanics: Chapter 3 in Bergfinnur Duurhus, Jan Philip Solovej, Mathematical Physics, lecture notes
`https://noter.math.ku.dk/mathphys2014.pdf`
- Formalism of quantum mechanics: Section 1 and 2 in Benjamin Schlein's lecture notes
`https://www.math.uzh.ch/ve-vo-det?key1=1831&key2=3282&keySemId=36&L=1`
- Overview: Stephen J. Gustafson, Israel Michael Sigal, Mathematical Concepts of Quantum Mechanics.
`https://www.math.utoronto.ca/~sigal/semlectnotes/1.pdf`
- Constructive approach to functional analysis: Elliott H. Lieb, Michael Loss, Analysis.
- Heuristic approach: Jun John Sakurai, Modern Quantum Mechanics

Plan of the course

The course will start the last week of February until the last week of May (30 hours). It will be **Tuesday at 16:00-18:00 and Thursday at 11:00-13:00**, and it will alternate a week with 2+2 hours and a week with 2 hours. The classes will be at the Dipartimento di Matematica (upon request also online, please send me an email if you want to receive the link).

Calendar

1. Week 24/02:
   - Tuesday 25/02 at 16:00-18:00 in Aula Seminari and online
   - Thursday at 11:00-13:00 in Aula Seminari and online
2. Week 03/03:
   - Tuesday 04/03 at 16:00-18:00 in Saletta Riunioni and online
3. Week 10/03:
   - Tuesday 11/03 at 16:00-18:00 in Aula Seminari and online
   - Thursday 13/03 at 11:00-13:00 in Sala Riunioni and online
4. Week 17/03:
   - Tuesday 18/03 at 16:00-18:00 in Aula Seminari CANCELLED! (rescheduled on 29/05)
5. Week 24/03:
   - Tuesday 25/03 at 16:00-18:00 in Sala Riunioni
   - Thursday 27/03 at 11:00-13:00 in Sala Riunioni

   Break of one week

6. Week 07/04:
   - Tuesday 08/04 at 16:00-18:00 in Sala Riunioni
   - Thursday 10/04 at 11:00-13:00 in Saletta Riunioni
7. week 14/04:
   - Tuesday 15/04 at 16:00-18:00 in Sala Riunioni

   Easter break

8. week 28/04:
   - Tuesday 29/04 at 16:00-18:00 in Sala Riunioni

   Break of two-weeks

9. week 19/05:
   - Tuesday 20/05 at 16:00-18:00 in Sala Riunioni
   - Thursday 22/05 at 11:00-13:00 in Sala Riunioni
10. week 26/05:
    - Tuesday 27/05 at 16:00-18:00 in Sala Riunioni
    - Thursday 29/05 at 11:00-13:00 in Sala Riunioni

Class Material

Notes will be uploaded on the Teams page of the course.