Mathematical Methods of Quantum Mechanics

PhD course in Mathematics @ Unipi

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Course description

Quantum mechanics is one of the central theories of physics, whose language describes a huge variety of systems, from elementary particles to superconducting materials and quantum computers. Fascinating mathematical problems arise in this context, posing hard challenges in functional analysis, partial differential equations and operator theory. This course is an introduction to the rigorous mathematical framework of quantum mechanics and functional analytic methods for the study of the Schrödinger equation.

The course consists of two parts. The first part (of approximately 20 hours) will start with an overview on the theory of linear operators on Hilbert spaces and self-adjointness. We will then study properties of Schrödinger operators and quantum dynamics. Some selected topics related to the physical motivations of quantum mechanics will be presented.

The second part (of approximately 10 hours) will be devoted to many-body quantum mechanics. We will introduce the mathematical questions and tools in this context and conclude with a discussion of current research topics.

Class requirements

No previous knowledge of quantum mechanics is expected, but a solid background in analysis will be necessary to have a deep undestanding of the topics of this course. The class is open to advanced master students.

Preliminary meeting

There will be a prelimiary meeting the first week of February (date and time TBA) where I will give a presentation of the course.

Program of the course

(may be subjected to small changes)

First part: the mathematical framework of quantum mechanics

1. The formalism of quantum mechanics, Banach spaces, L^p-spaces, bounded operators, Hilbert spaces, Fréchet-Riesz representation theorem, closed graph theorem, densely defined operators. Axioms of Quantum Mechanics.

2. *Why quantum mechanics? Stern-Gerlach experiment. Description of the spin.*

3. Resolvent, spectrum, analytic functions, operator-valued analytic functions, Neumann series.

4. Adjoint operator, symmetric and self-adjoint operators, Hellinger-Töplitz theorem, momentum operator, generalized theorem of the bounded inverse, self-adjointness criterion
5. *Harmonic oscillator*
6. Kato-Rellich theorem, Schrödinger equation, essential self-adjointness, Schrödinger operators, unitary equivalence, Fourier transform.
7. Multiplication operators, Weyl criterion, Laplace operators, Sobolev inequalities.
8. *Bell inequality, EPR paradox.*
9. Uniform boundedness principle, strong convergence, strongly continuous unitary groups and generators.
10. Symmetries, quantum Noether theorem.
11. *Discussion of Exercise sheets.*

Second part: many-body quantum mechanics.

1. Confined systems, thermodynamic limit, tensor product of Hilbert spaces, unitary representation of the symmetric group, symmetric and antisymmetric tensor product. Compact operators, trace-class and Hilbert-Schmidt operators, integral kernels.
2. Hamiltonians with pair interaction, Fock space, creation and annihilation operators, canonical (anti)commutation relations
3. Second quantization, reduced density matrix, Bose-Einstein condensation
4. Bogoliubov theory/derivation of the nonlinear Schrödinger equation

References

Main references

- Gerald Teschl: Mathematical Methods in Quantum Mechanics, With Applications to Schrödinger Operators
`https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/index.html`
- Jan Philip Solovej: Many Body Quantum Mechanics, Draft of Lecture Notes of March 5, 2014
`https://web.math.ku.dk/~solovej/MANYBODY/mbnotes-ptn-5-3-14.pdf`

Other references

- Hamiltonian mechanics: Chapter 3 in Bergfinnur Duurhus, Jan Philip Solovej, Mathematical Physics, lecture notes
`https://noter.math.ku.dk/mathphys2014.pdf`
- Overview: Stephen J. Gustafson, Israel Michael Sigal, Mathematical Concepts of Quantum Mechanics.
`https://www.math.utoronto.ca/~sigal/semlectnotes/1.pdf`
- Constructive approach to functional analysis: Elliott H. Lieb, Michael Loss, Analysis.
- Heuristic approach: Jun John Sakurai, Modern Quantum Mechanics

Plan of the course

The course will start the last week of February until the last week of May (30 hours). Approximately, it will alternate a week with 2+2 hours and a week with 2 hours.

Calendar (Preliminary)

1. week 24/02: 2+2 hours
2. week 03/03: 2 hours
3. week 10/03: 2+2 hours
4. week 17/03: 2 hours
5. week 24/03: 2+2 hours

   Break of one week

6. week 07/04: 2+2 hours
7. week 14/04: 2 hours

   Easter break

8. week 28/04: 2 hours

   Break of two-weeks

9. week 19/05: 2+2 hours
10. week 26/05: 2 hours